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A NOTE ON DEGENERATE STIRLING POLYNOMIALS OF THE SECOND KIND

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Abstract. In this paper, we consider the degenerate Stirling polynomials of the second kind which are derived from the generating function. In addition, we give some new identities for these polynomials.

1. Introduction

For $n \in \mathbb{N} \cup \{0\}$, as is well known, the Stirling number of the first kind is defined by

$$(x)_0 = 1, \ (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \geq 1). \tag{1.1}$$

Note that

$$S_1(n+1,k) = S_1(n,k-1) - nS_1(n,k), \ (1 \leq k \leq n), \quad \text{(see [6])}. \tag{1.2}$$

The Stirling number of the second kind is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \ (n \geq 0), \quad \text{(see [1, 2, 4])}. \tag{1.3}$$

From (1.3), we note that

$$S_2(n+1,k) = kS_2(n,k) + S_2(n,k-1), \tag{1.4}$$

where $1 \leq k \leq n$, (see [2,3,4,5]). The generating functions for $S_1(n,k)$ and $S_2(n,k)$, $(n,k \geq 0)$, are given by

$$\frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \tag{1.5}$$

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and

$$\frac{1}{k!}\left(e^{t}-1\right)^{k}=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\quad \text{(see [9])}. \tag{1.6}$$

Now, we define the difference operator Δ as follows:

$$\Delta f(x)=f(x+1)-f(x),\quad \text{(see [3,7])}. \tag{1.7}$$

From (1.7), we have

$$\Delta^n f(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(x+k),\ (n\in\mathbb{N}\cup\{0\}). \tag{1.8}$$

and

$$f(x)\approx\sum_{k=0}^{\infty}\binom{x}{k}\Delta^k f(0),\quad \text{(see [3,7])}. \tag{1.9}$$

From (1.8), we note that

$$\Delta^k 0^n=\begin{cases}S_2(n,k) & \text{if } n\geq k\\ 0 & \text{if } n<k.\end{cases} \tag{1.10}$$

The Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad \text{(see [1-10])}. \tag{1.11}$$

When $x=0$, $B_n=B_n(0)$, $(n\geq 0)$, are called the Bernoulli numbers.

The Euler polynomials are defined by the generating function as follows:

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\quad \text{(see [1,2])}. \tag{1.12}$$

When $x=0$, $E_n=E_n(0)$ are Euler numbers. We observe that

$$\begin{aligned}\frac{2}{e^t+1}&=\left(\frac{e^t-1}{2}+1\right)^{-1}=\sum_{l=0}^{\infty}\left(\frac{e^t-1}{2}\right)^{l}(-1)^l\\ &=\sum_{l=0}^{\infty}(-1)^l 2^{-l}l!\sum_{n=l}^{\infty}S_2(n,l)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_2(n,l)(-1)^l 2^{-l}l!\right)\frac{t^n}{n!}.\end{aligned} \tag{1.13}$$

By (1.12) and (1.13), we get

$$E_n = \sum_{l=0}^{n} S_2(n,l) 2^{-l} l! (-1)^l, \ (n \geq 0).$$

In [1], L. Carlitz consider the degenerate Bernoulli and Euler polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, \tag{1.14}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.15}$$

Note that

$$\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x), \ \lim_{\lambda \to 0} \mathcal{E}_{n,\lambda}(x) = E_n(x), \ (n \geq 0).$$

In this paper, in the viewpoint (1.14) and (1.15), we consider the degenerate Stirling polynomials of the second kind which are derived from the generating function. In addition, we give some new identities for these polynomials.

2. Degenerate Stirling polynomials of the second kind

Now, we define the Stirling polynomials of the second kind which are given by the generating function to be

$$\frac{1}{k!} e^{xt} \left(e^t - 1\right)^k = \sum_{n=k}^{\infty} S_2(n,k|x) \frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\begin{aligned} \frac{1}{k!} e^{xt} \left(e^t - 1\right)^k &= \frac{1}{k!} \left(e^t - 1\right)^k e^{xt} \\ &= \left(\sum_{l=k}^{\infty} S_2(l,k) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!} x^m t^m \right) \\ &= \sum_{n=k}^{\infty} \left\{ \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) x^{n-l} \right\} \frac{t^n}{n!} \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$S_2(n,k|x) = \sum_{l=k}^{n} \binom{n}{l} S_2(l,k) x^{n-l}, \ (n,k \geq 0). \tag{2.3}$$

When $x = 0$, we easily get $S_2(n,k|0) = S_2(n,k)$. Now, we consider the degenerate Stirling polynomials which are defined by the generating function as follows:

$$\frac{1}{k!}(1+\lambda t)^{\frac{x}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k|x)\frac{t^n}{n!}. \tag{2.4}$$

Now, we observe that

$$\begin{aligned}(1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{l=0}^{\infty}\binom{\frac{x}{\lambda}}{l}\lambda^l t^l = \sum_{l=0}^{\infty}\left(\frac{x}{\lambda}\right)_l \lambda^l \frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!} = \sum_{l=0}^{\infty}\binom{x}{l}_{\lambda} t^l,\end{aligned} \tag{2.5}$$

where

$$(x)_{0,\lambda} = 1,\ (x)_{l,\lambda} = x(x-\lambda)\cdots(x-(l-1)\lambda),\ (l \geq 1), \tag{2.6}$$

and

$$\binom{x}{l}_{\lambda} = \frac{(x)_{l,\lambda}}{l!} = \frac{x(x-\lambda)\cdots(x-(l-1)\lambda)}{l!}. \tag{2.7}$$

For $n \in \mathbb{N}$, let us define λ-analogue of $n!$ as follows:

$$\begin{aligned}(n)_{\lambda}! &= n(n-\lambda)(n-2\lambda)\cdots(n-(n-1)\lambda) \\ &= (n)_{n,\lambda},\end{aligned} \tag{2.8}$$

and

$$\binom{n}{k}_{\lambda} = \frac{(n)_{\lambda}!}{k!(n-k\lambda)_{n-k,\lambda}} = \frac{(n)_{k,\lambda}}{k!},\ (n \geq k \geq 0). \tag{2.9}$$

From (2.4), we have

$$\begin{aligned}&\frac{1}{k!}(1+\lambda t)^{\frac{x}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k \\ &= \frac{1}{k!}(1+\lambda t)^{\frac{x}{\lambda}}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1+\lambda t)^{\frac{l}{\lambda}} \\ &= \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1+\lambda t)^{\frac{l+x}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(\frac{n!}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\binom{l+x}{n}_{\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

Therefore, by (2.4) and (2.10), we obtain the following theorem.

Theorem 2.1. *For $n, k \geq 0$, we have*

$$\frac{n!}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\binom{l+x}{n}_{\lambda}=\begin{cases}S_{2,\lambda}(n,k|x), & \text{if } n\geq k\\ 0, & \text{if } n<k.\end{cases}$$

From (2.4), we have

$$\begin{aligned}
&\frac{1}{k!}(1+\lambda t)^{\frac{x}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^{k}\\
&=\frac{1}{k!}e^{\frac{x}{\lambda}\log(1+\lambda t)}\left(e^{\frac{1}{\lambda}\log(1+\lambda t)}-1\right)^{k}\\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{\frac{x+l}{\lambda}\log(1+\lambda t)}\\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{m=0}^{\infty}\left(\frac{x+l}{\lambda}\right)^{m}\frac{1}{m!}\big(\log(1+\lambda t)\big)^{m}\\
&=\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{m=0}^{\infty}\frac{(x+l)^{m}}{\lambda^{m}}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^{n}t^{n}}{n!} \qquad (2.11)\\
&=\sum_{n=0}^{\infty}\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)(x+l)^{m}\frac{t^{n}}{n!}\\
&=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(x+l)^{m}\right)\lambda^{n-m}S_1(n,m)\right\}\frac{t^{n}}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{k!}\Delta^{k}x^{m}\lambda^{n-m}S_1(n,m)\right)\frac{t^{n}}{n!}.
\end{aligned}$$

Therefore, by (2.4) and (2.11), we obtain the following theorem.

Theorem 2.2. *For $n, k \geq 0$, we have*

$$\sum_{m=0}^{n}\frac{1}{k!}\Delta^{k}x^{m}\lambda^{n-m}S_1(n,m)\begin{cases}S_{2,\lambda}(n,k|x), & \text{if } n\geq k\\ 0, & \text{if } n<k.\end{cases}$$

By (1.8), we easily get

$$\begin{aligned}\Delta^k x^{m+1} &= \sum_{l=0}^{k}\binom{k}{l}(x+l)^{m+1}(-1)^{k-l} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(x+l)^m(x+l)\\ &= x\sum_{l=0}^{k}\binom{k}{l}(x+l)^m(-1)^{k-l} + \sum_{l=0}^{k}\binom{k}{l}(x+l)^m l(-1)^{k-l}\\ &= x\Delta^k x^m + k\sum_{l=1}^{k}\binom{k-1}{l-1}(x+l)^m(-1)^{k-l}\\ &= x\Delta^k x^m + k\sum_{l=1}^{k}\left\{\binom{k}{l}-\binom{k-1}{l}\right\}(x+l)^m(-1)^{k-l}\\ &= x\Delta^k x^m + k\sum_{l=0}^{k}\left\{\binom{k}{l}-\binom{k-1}{l}\right\}(x+l)^m(-1)^{k-l}\\ &= x\Delta^k x^m + k\big(\Delta^k x^m + \Delta^{k-1}x^m\big).\end{aligned} \tag{2.12}$$

From (1.2) and (2.12), we have

$$\begin{aligned}S_{2,\lambda}(n+1,k|x) &= \sum_{m=0}^{n+1}\frac{1}{k!}\Delta^k x^m\lambda^{n+1-m}S_1(n+1,m)\\ &= \sum_{m=1}^{n+1}\frac{1}{k!}\Delta^k x^m\lambda^{n+1-m}S_1(n+1,m)\\ &= \sum_{m=1}^{n+1}\frac{1}{k!}\Delta^k x^m\lambda^{n+1-m}\big(S_1(n,m-1)-nS_1(n,m)\big)\\ &= \sum_{m=1}^{n+1}\frac{1}{k!}\Delta^k x^m\lambda^{n+1-m}S_1(n,m-1)\\ &\quad - n\lambda\sum_{m=0}^{n}\frac{1}{k!}\Delta^k x^m\lambda^{n-m}S_1(n,m)\\ &= \sum_{m=1}^{n+1}\frac{1}{k!}\Delta^k x^m\lambda^{n+1-m}S_1(n,m-1) - n\lambda S_{2,\lambda}(n,k|x).\end{aligned} \tag{2.13}$$

Now, we observe that

$$\begin{aligned}
&\sum_{m=1}^{n+1}\frac{1}{k!}\Delta^k x^m\lambda^{n+1-m}S_1(n,m-1)=\sum_{m=0}^{n}\frac{1}{k!}\Delta^k x^{m+1}\lambda^{n-m}S_1(n,m)\\
&=\sum_{m=0}^{n}\frac{1}{k!}\left\{x\Delta^k x^m+k\big(\Delta^k x^m+\Delta^{k-1}x^m\big)\right\}\lambda^{n-m}S_1(n,m)\\
&=x\frac{1}{k!}\sum_{m=0}^{n}\Delta^k x^m\lambda^{n-m}S_1(n,m)+k\frac{1}{k!}\Delta^k x^m\lambda^{n-m}S_1(n,m)\\
&+\frac{1}{(k-1)!}\sum_{m=0}^{n}\Delta^{k-1}x^m\lambda^{n-m}S_1(n,m)\\
&=(x+k)S_{2,\lambda}(n,k|x)+S_{2,\lambda}(n,k-1|x),\ (1\le k\le n).
\end{aligned}\tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.3. *For $1\le k\le n$, we have*

$$S_{2,\lambda}(n+1,k|x)=(x+k)S_{2,\lambda}(n,k|x)+S_{2,\lambda}(n,k-1|x)-n\lambda S_{2,\lambda}(n,k|x).$$

Note that

$$\lim_{\lambda\to 0}S_{2,\lambda}(n+1,k|x)=(x+k)S_2(n,k|x)+S_2(n,k-1|x).\tag{2.15}$$

Thus, by (2.15) we get

$$S_2(n+1,k|x)=(x+k)S_2(n,k|x)+S_2(n,k-1|x),\tag{2.16}$$

where $1\le k\le n$. As is known, the higher-order Carlitz degenerate Euler polynomials are defined by the generating function as follows:

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\ (r\in\mathbb{N}).\tag{2.17}$$

From (2.17), we note that

$$
\begin{aligned}
&\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}} = 2^{r}\big((1+\lambda t)^{\frac{1}{\lambda}}+1\big)^{-r}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{2}+1\right)^{-r}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \sum_{l=0}^{\infty}\binom{r+l-1}{l}2^{-l}(-1)^{l}\big((1+\lambda t)^{\frac{1}{\lambda}}-1\big)^{l}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \sum_{l=0}^{\infty}\binom{r+l-1}{l}2^{-l}(-1)^{l}l!\sum_{n=l}^{\infty}S_{2,\lambda}(n,l|x)\frac{t^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{r+l-1}{l}2^{-l}(-1)^{l}l!S_{2,\lambda}(n,l|x)\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{2.18}
$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$
\mathcal{E}_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\binom{r+l-1}{l}2^{-l}(-1)^{l}l!S_{2,\lambda}(n,l|x).
$$

For $r \in \mathbb{N}$, the r-Whitney numbers of the second kind $W_{m,r}(n,k)$ are defined by

$$
(mx+r)^{n} = \sum_{k=0}^{n}m^{k}W_{m,r}(n,k)(x)_{k}, \quad \text{(see [8]).} \tag{2.19}
$$

where $n, r \in \mathbb{N}\cup\{0\}$ and $m \in \mathbb{N}$. The generating function of Whitney numbers is given by

$$
\frac{1}{m^{k}k!}e^{rt}(e^{mt}-1)^{k} = \sum_{n=k}^{\infty}W_{m,r}(n,k)\frac{t^{n}}{n!}. \tag{2.20}
$$

From (2.20), we note that

$$
\begin{aligned}
\frac{1}{m^{k}k!}e^{rt}(e^{mt}-1)^{k} &= \frac{1}{m^{k}}\left(\sum_{l=0}^{\infty}r^{l}\frac{t^{l}}{l!}\right)\left(\sum_{i=k}^{\infty}S_{2}(i,k)\frac{m^{i}t^{i}}{i!}\right) \\
&= \frac{1}{m^{k}}\sum_{n=k}^{\infty}\left(\sum_{i=k}^{n}S_{2}(i,k)\binom{n}{i}r^{n-i}m^{i}\right)\frac{t^{n}}{n!}.
\end{aligned}
\tag{2.21}
$$

Thus, by (2.20) and (2.21), we get

$$W_{m,r}(n,k)=\sum_{i=k}^{n}\binom{n}{i}r^{n-i}S_2(i,k)m^{i-k}. \tag{2.22}$$

By (2.19), we easily get

$$(mx+r)^n=\sum_{k=0}^{n}\left(\sum_{i=k}^{n}\binom{n}{i}r^{n-i}m^iS_2(i,k)\right)(x)_k, \tag{2.23}$$

where $n,r\in\mathbb{N}\cup\{0\}$ and $m\in\mathbb{N}$.

Note that $W_{1,0}(n,k)=S_2(n,k)$ and $W_{1,r}(n,k)=S_2(n,k|r)$. From (2.20), we note that

$$\begin{aligned}
&\frac{1}{m^kk!}e^{rt}\left(e^{mt}-1\right)^k=\frac{1}{m^kk!}e^{rt}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lm}\\
&=\frac{1}{m^kk!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{(lm+r)t}=\frac{1}{m^kk!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{n=0}^{\infty}(lm+r)^n\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{m^kk!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(lm+r)^n\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}m^{n-k}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\left(l+\tfrac{r}{m}\right)^n\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(m^{n-k}\frac{1}{k!}\Delta^k\left(\frac{r}{m}\right)^n\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.24}$$

Therefore, by (2.20) and (2.24), we obtain the following theorem.

Theorem 2.5. *For $m\in\mathbb{N}$ and $n,k,r\in\mathbb{N}\cup\{0\}$, we have*

$$W_{m,r}(n,k)=\begin{cases}m^{n-k}\frac{1}{k!}\Delta^k\left(\frac{r}{m}\right)^n, & \text{if } n\geq k,\\ 0, & \text{if } n<k.\end{cases}$$

For $1\leq k\leq n$, by Theorem 5, we easily get

$$W_{m,r}(n+1,k)=(r+mk)W_{m,r}(n,k)+W_{m,r}(n,k-1). \tag{2.25}$$

We consider the degenerate Whitney numbers which are defined by the generating function to be

$$\frac{1}{m^kk!}(1+\lambda t)^{\frac{r}{\lambda}}\left((1+\lambda t)^{\frac{m}{\lambda}}-1\right)^k=\sum_{n=k}^{\infty}W_{m,r}(n,k|\lambda)\frac{t^n}{n!}, \tag{2.26}$$

where $m \in \mathbb{N}$, $n,k,r \in \mathbb{N} \cup \{0\}$. Note that $\lim_{\lambda \to 0} W_{m,r}(n,k|\lambda) = W_{m,r}(n,k)$. Now, we observe that

$$\begin{aligned} &\frac{1}{m^k k!}(1+\lambda t)^{\frac{r}{\lambda}}\big((1+\lambda t)^{\frac{m}{\lambda}}-1\big)^k \\ &= \frac{1}{m^k k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(1+\lambda t)^{\frac{ml}{\lambda}}(1+\lambda t)^{\frac{r}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(\frac{n!}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\binom{ml+r}{n}_{\lambda}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.6. *For $m \in \mathbb{N}$, and $n,k,r \in \mathbb{N}\cup\{0\}$, we have*

$$\frac{n!}{k!}\sum_{l=0}^{k}\binom{k}{l}\binom{ml+r}{n}_{\lambda}(-1)^{k-l} = \begin{cases} W_{m,r}(n,k|\lambda), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

From (2.26), we note that

$$\begin{aligned} &\frac{1}{m^k k!}(1+\lambda t)^{\frac{r}{\lambda}}\big((1+\lambda t)^{\frac{m}{\lambda}}-1\big)^k = \frac{1}{m^k k!}e^{\frac{r}{\lambda}\log(1+\lambda t)}\big(e^{\frac{m}{\lambda}\log(1+\lambda t)}-1\big)^k \\ &= \frac{1}{k! m^k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{\frac{lm+r}{\lambda}\log(1+\lambda t)} \\ &= \frac{1}{k! m^k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{j=0}^{\infty}\left(\frac{lm+r}{\lambda}\right)^j\frac{1}{j!}\big(\log(1+\lambda t)\big)^j \\ &= \frac{1}{k! m^k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\sum_{j=0}^{\infty}\left(\frac{lm+r}{\lambda}\right)^j\sum_{n=j}^{\infty}S_1(n,j)\frac{\lambda^n t^n}{n!} \\ &= \frac{1}{k! m^k}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\lambda^{n-j}S_1(n,j)\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}(lm+r)^j\right)\frac{t^n}{n!} \\ &= \frac{1}{k! m^k}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\lambda^{n-j}m^j S_1(n,j)\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\big(l+\tfrac{r}{m}\big)^j\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\lambda^{n-j}m^j S_1(n,j)\frac{1}{k! m^k}\Delta^k\left(\frac{r}{m}\right)^j\right)\frac{t^n}{n!}. \end{aligned} \tag{2.28}$$

Therefore, by (1.4) and (2.28), we obtain the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}$, and $n,k,r \in \mathbb{N} \cup \{0\}$, we have*

$$\frac{1}{k!m^k}\sum_{j=0}^{n}\lambda^{n-j}S_1(n,j)m^j\Delta^k\left(\frac{r}{m}\right)^j = \begin{cases} W_{m,r}(n,k|\lambda), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

For $1 \leq k \leq n$, by Theorem 7, we get

$$\begin{aligned}
W_{m,r}(n+1,k|\lambda) &= \frac{1}{k!m^k}\sum_{j=0}^{n+1}\lambda^{n+1-j}S_1(n+1,j)m^j\Delta^k\left(\frac{r}{m}\right)^j \\
&= \frac{1}{k!m^k}\sum_{j=1}^{n+1}\lambda^{n+1-j}\Big\{S_1(n,j-1)-nS_1(n,j)\Big\}m^j\Delta^k\left(\frac{r}{m}\right)^j \\
&= \frac{1}{k!m^k}\sum_{j=1}^{n+1}\lambda^{n+1-j}S_1(n,j-1)m^j\Delta^k\left(\frac{r}{m}\right)^j - \frac{n\lambda}{k!m^k}\sum_{j=1}^{n+1}\lambda^{n-j}S_1(n,j)m^j\Delta^k\left(\frac{r}{m}\right)^j \\
&= \frac{1}{k!m^k}\sum_{j=0}^{n}\lambda^{n-j}S_1(n,j-1)m^{j+1}\Delta^k\left(\frac{r}{m}\right)^{j+1} \\
&\quad - \frac{n\lambda}{k!m^k}\sum_{j=0}^{n}\lambda^{n-j}S_1(n,j)m^j\Delta^k\left(\frac{r}{m}\right)^j \\
&= \frac{m}{k!m^k}\sum_{j=0}^{n}\lambda^{n-j}S_1(n,j)m^j\Delta^k\left(\frac{r}{m}\right)^{j+1} - n\lambda W_{m,r}(n,k|\lambda).
\end{aligned} \tag{2.29}$$

Now, we observe that

$$\begin{aligned}
\Delta^k\left(\frac{r}{m}\right)^{j+1} &= \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^{j+1} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^j\Big(l+\frac{r}{m}\Big) \\
&= \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^j l + \frac{r}{m}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^j \\
&= \frac{r}{m}\Delta^k\left(\frac{r}{m}\right)^j + k\sum_{l=1}^{k}\binom{k-1}{l-1}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^j \\
&= \frac{r}{m}\Delta^k\left(\frac{r}{m}\right)^j + k\sum_{l=0}^{k}\left\{\binom{k}{l}-\binom{k-1}{l}\right\}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^j
\end{aligned}$$

$$= \frac{r}{m}\Delta^k\left(\frac{r}{m}\right)^j + k\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\Big(l+\frac{r}{m}\Big)^j + k\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}\Big(l+\frac{r}{m}\Big)^j$$
$$= \frac{r}{m}\Delta^k\left(\frac{r}{m}\right)^j + k\Delta^k\left(\frac{r}{m}\right)^j + k\Delta^{k-1}\left(\frac{r}{m}\right)^j. \tag{2.30}$$

From (2.29) and (2.30), we note that

$$W_{m,r}(n+1,k|\lambda) = \frac{m}{k!m^k}\sum_{j=0}^{m} m^j S_1(n,j)\lambda^{n-j}\Delta^k\left(\frac{r}{m}\right)^{j+1} - n\lambda W_{m,r}(n,k|\lambda)$$
$$= \frac{m}{k!m^k}\sum_{j=0}^{m} m^j S_1(n,j)\lambda^{n-j}\left\{\frac{r}{m}\Delta^m\left(\frac{r}{m}\right)^j + k\Delta^k\left(\frac{r}{m}\right)^j + k\Delta^{k-1}\left(\frac{r}{m}\right)^j\right\}$$
$$- n\lambda W_{m,r}(n,k|\lambda)$$
$$= \frac{r}{k!m^k}\sum_{j=0}^{m} m^j S_1(n,j)\lambda^{n-j}\Delta^m\left(\frac{r}{m}\right)^j + \frac{mk}{k!m^k}\sum_{j=0}^{m} m^j S_1(n,j)\lambda^{n-j}\Delta^k\left(\frac{r}{m}\right)^j$$
$$+ \frac{1}{(k-1)!m^{k-1}}\sum_{j=0}^{m} m^j S_1(n,j)\lambda^{n-j}\Delta^{k-1}\left(\frac{r}{m}\right)^j - n\lambda W_{m,r}(n,k|\lambda)$$
$$= rW_{m,r}(n,k|\lambda) + mkW_{m,r}(n,k|\lambda) + W_{m,r}(n,k-1|\lambda) - n\lambda W_{m,r}(n,k|\lambda).$$

Therefore, we obtain the following theorem.

Theorem 2.8. *For $1 \le k \le n$, we have*

$$W_{m,r}(n+1,k|\lambda) = (r+mk)W_{m,r}(n,k|\lambda) + W_{m,r}(n,k-1|\lambda) - n\lambda W_{m,r}(n,k|\lambda).$$

Remark. From (2.5), we note that

$$(1+\lambda t)^{\frac{x+y}{\lambda}} = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{y}{\lambda}} = \left(\sum_{l=0}^{\infty}\binom{x}{l}_\lambda t^l\right)\left(\sum_{m=0}^{\infty}\binom{y}{m}_\lambda t^m\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{y}{m}_\lambda\binom{x}{n-m}_\lambda\right)t^n. \tag{2.31}$$

By (2.31), we easily get

$$(1+\lambda t)^{\frac{x+y}{\lambda}} = \sum_{n=0}^{\infty}\binom{x+y}{n}_\lambda t^n. \tag{2.32}$$

Comparing the coefficients on the both sides of (2.31) and (2.32), we have

$$\sum_{m=0}^{n}\binom{y}{m}_\lambda\binom{x}{n-m}_\lambda = \binom{x+y}{n}_\lambda,\ (n \ge 0).$$

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